

Indirect Rate-Distortion Function of a Binary i.i.d Source

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Abstract

The indirect source-coding problem in which a Bernoulli process is compressed in a lossy manner from its noisy observations is considered. These noisy observations are obtained by passing the source sequence through a binary symmetric channel so that the channel crossover probability controls the amount of information available about the source realization at the encoder. We use classic results in rate-distortion theory to compute an expression of the rate-distortion function for this model, where the Bernoulli source is not necessarily symmetric. The indirect rate-distortion function is given in terms of a solution to a simple equation. In addition, we derive an upper bound on the indirect rate-distortion function which is given in a closed. These expressions capture precisely the expected behavior that the noisier the observations, the smaller the return from increasing bit-rate to reduce distortion.

Index Terms

Indirect rate distortion problem; Binary source; Binary symmetric channel;

I. INTRODUCTION

The optimal trade-off between bit-rate and average distortion in the representation of an information source is given by the Rate-Distortion Function (RDF): the RDF provides the minimum rate necessary to describe a source when its reconstruction is allowed to be to within a given average distortion from the original sequence. A natural extension of this source coding problem is the scenario in which the encoder cannot observe the source directly but obtains only noisy observations. This could be due to a number of phenomena such as environmental noise, finite precision quantization and sub-sampling [1]. In this setup, the encoder is required to describe the source from another process statistically correlated with the source itself: this problem is known as *indirect* or *remote* source coding [2, Sec. 3.5].

An interesting motivation for the indirect source coding problem arises in centralized sensing networks in which each sensor in the network is required to transmit its observation to a remote processing unit. Restrictions on the computational complexity and power consumption of the sensors make local processing infeasible and thus the uncompressed data has to be communicated over the network. The communication toward the central unit introduces noise in the sensors' observations and the compression rate of the data acquired at the central node is determined by the indirect RDF.

The general structure of an indirect source coding problem is depicted in Figure 1: the source process, X^n , is passed through the noisy channel $P_{Y|X}^n$ to obtain the signal Y^n . The encoder compresses the sequence Y^n at rate R and the compressed observation is provided noiselessly to the decoder. The receiver produces the sequence \hat{X}^n which is a reconstruction of the original signal X^n to within a prescribed average distortion.

While in the direct source coding problem the RDF describes the optimal trade-off between the code rate R and distortion D , another quantity of merit in the indirect problem is the channel $P_{Y|X}$. By characterizing the trade-off in the indirect problem, namely by an *indirect RDF*, it is possible to study the effect of the channel quality on the optimal rate-distortion trade-off. For instance, it is of interest to characterize the amount of additional code-rate needed to maintain a fixed distortion level as the observations become noisier.

It has long been noticed [2], [3], [4] that an indirect source coding problem can be reduced to a standard source coding problem by the following argument: it is possible to consider the observable process Y^n as the source in the standard source coding problem by amending the fidelity criterion to capture the distance between the reconstructed symbol \hat{X}^n and all possible realizations of the original source realization X^n weighed according to the probability of their appearance given Y^n . A particularly intuitive form of this observation appears in the case of a quadratic distortion, where the amended fidelity criterion can be decomposed as the sum of two terms: (i) the mean squared error (MSE) estimation of the source from its observation plus (ii) the error in describing the MSE estimate under a rate-limited description [4]. This separation allows one to obtain the closed form expression of the indirect RDF in the Gaussian source, quadratic distortion and additive Gaussian noise case [5], [1].

While, in general, similar separation results for other models do not exist, it may still be possible to solve the direct problem using the amended distortion measure. This approach is explored in this paper for the important case of a binary i.i.d source, bit flipping noise and the Hamming distortion.

Related Work: The source coding problem was first introduced by Shannon in [6] while he provided the first of the source coding theorem in [7]. Indirect rate-distortion problem was first introduced by Dobrushin and Tsybakov in [5]. The authors of

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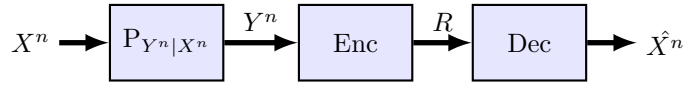


Fig. 1: Indirect source coding model

[5] derived a closed form solution for the indirect RDF in the Gaussian stationary case and, implicitly, showed an equivalence of the indirect problem to a direct source coding problem with an amended fidelity criterion. Wolf and Ziv [4] showed that, in the case of a quadratic distortion, the new fidelity criterion identified in [3] decomposes into the sum of two terms, only one of which depends on the source coding rate R . Wolf and Ziv also computed the indirect RDF (iRDF) in various cases include the case of a Bernoulli source observed through a binary symmetric channel under quadratic distortion. Since the quadratic distortion of a binary sequence is not larger than its Hamming distance, their result provides a lower bound on the iRDF of the same source under the Hamming distance considered in this work. Berger [2] noted the equivalence of the indirect problem to a modified direct problem with a new fidelity criterion, and gave an interpretation of the new fidelity criterion as the conditional expectation of the the original distortion measure given the source and noise realizations.

In the special case of a Bernoulli observed through a binary channel, the computation of the iRDF is greatly simplified when the source is symmetric $\mathbb{P}(X_n = 1) = 1/2$ [2, Exc. 3.8]. In our setting where the observation are given by a binary symmetric channel, this iRDF is given by

$$R_{X|Y}(D) = \begin{cases} 1 - h\left(\frac{D-p}{1-2p}\right) & p < D < 1/2 \\ 0 & D \geq 1/2, \end{cases} \quad (1)$$

where $h(x)$ is the binary entropy function and $p < 1/2$ and $\bar{p} = 1 - p$ (the case $p > 1/2$ can be treated in a similar fashion). The symmetric case can also be obtained as a special case from [8] where indirect versions of the multiterminal setting of Slepian-Wolf and Wyner-Ziv problems were considered.

Contributions: We derive an expression for the iRDF of a Bernoulli process X^n with $\mathbb{P}(X_n = 1) = \alpha$ given its observation Y^n through a binary symmetric channel with crossover probability p for the general case of $\alpha \in [0, 1/2)$. This iRDF $R_{X|Y}(D)$ is obtained by finding the unique root to an equation whose parameters are determined by α , p and D . Additionally, we show that an upper bound on $R_{X|Y}(D)$ expressed as (for $p \leq 0.5$)

$$\bar{R}_{X|Y}(D) = \begin{cases} h(\alpha \star p) - h\left(\frac{D-p}{1-2p}\right) & p \leq D \leq 1/2 \\ 0 & D \geq \alpha, \end{cases} \quad (2)$$

where $\alpha \star p \triangleq p\bar{\alpha} + \alpha\bar{p}$ with equality if and only if $\alpha = 1/2$, in which case $R_{X|Y}(D) = \bar{R}_{X|Y}(D)$ for all D .

The rest of this paper is organized as follows: the indirect source coding problem and the relevant background literature are introduced in Sec. II. The main results are derived in Sec. III. Finally, Sec. IV concludes the paper.

II. PROBLEM STATEMENT

We consider the indirect source coding problem depicted in Fig. 1: an encoder observes the discrete time process X^n through the noisy channel $P_{Y^n|X^n}$ and produces a sequence of coded symbols at rate R . From this sequence of coded symbols, the decoder produces a reconstructed sequence \hat{X}^n which must be to within maximum average distortion from X^n for a prescribed fidelity criterion.

More specifically, given a source sequence $X^n \triangleq \{X_k, k = 1, 2, \dots, n\}$ with alphabet \mathcal{X}^n , the encoder is provided with the sequence Y^n with alphabet \mathcal{Y}^n obtained from X^n through the channel $P_{Y^n|X^n}(Y^n|X^n)$ and maps this sequence unto the set $\{1 \dots 2^{\lfloor Rn \rfloor}\}$ through the mapping

$$W(Y^n) : \mathcal{Y}^n \rightarrow \{1 \dots 2^{\lfloor Rn \rfloor}\}. \quad (3)$$

The value $W(Y^n)$ is noiselessly communicated to the receiver which, in turns, produces the sequence \hat{X}^n with alphabet $\hat{\mathcal{X}}^n$ through the mapping

$$\hat{X}^n(W) : \{1 \dots 2^{\lfloor Rn \rfloor}\} \rightarrow \hat{\mathcal{X}}^n. \quad (4)$$

The sequence \hat{X}^n must be to within a distortion D from X^n for some chosen fidelity criterion $d_n(x^n, \hat{x}^n)$ which is measured with the per-letter distortion function $d(x_i, \hat{x}_i)$, as

$$d_n(x^n, \hat{x}^n) \triangleq \sum_{i=1}^n d(x_i, \hat{x}_i), \quad (5)$$

for some real-valued, bounded function $d(\cdot, \cdot)$.

The operational indirect RDF $\tilde{R}_{X|Y}(D)$ is defined as the minimal rate R in (3) and (4) such that the average distortion between X^n and \hat{X}^n in (5) does not exceed D , as the block-length n goes to infinity.

The indirect (Shannon's) RDF (iRDF) for the channel $P_{Y^n|X^n}$ is defined as

$$R_{X|Y}(D) = \liminf_{n \rightarrow \infty} R_n(D),$$

where

$$R_n(D) = \inf \frac{1}{n} I(Y^n; \hat{X}^n) \leq R,$$

and the infimum is taken over all mappings $Y^n \rightarrow \hat{X}^n = (3) \circ (4)$ such that the average distortion between X^n and \hat{X}^n is at most D .

The customary source coding problem [7], also *direct* source coding problem, is obtained from the indirect source coding problem by simply letting $Y^N = X^N$. It is noted in [3] that the problem of finding the operational indirect source coding rate $\tilde{R}_{X|Y}(D)$ can be reduced to a direct source coding problem for the observable process Y^n and a different distortion measure $\hat{d}(\cdot, \cdot)$ defined as

$$\hat{d}_n(y^n, \hat{x}^n) \triangleq \mathbb{E}[d_n(X^n, \hat{x}^n) | Y^n = y^n]. \quad (6)$$

Note that $\hat{d}(\cdot, \cdot)$ depends only on $d(\cdot, \cdot)$ and $P_{Y^n|X^n}$, which are determined by the structure of the original indirect rate distortion problem.

Since

$$\mathbb{E}[d_n(X^n, \hat{X}^n)] = \mathbb{E}[\hat{d}_n(Y, \hat{X})],$$

it follows that $R_{X|Y}(D)$ equals the (direct) RDF $R_Y(D)$ of the process Y^N under the fidelity criterion $\hat{d}(\cdot, \cdot)$. Shannon's source coding theorem [7] now implies

$$\tilde{R}_{X|Y}(D) = R_Y(D) = R_{X|Y}(D). \quad (7)$$

The reduction of the indirect source coding problem to a direct problem under $\hat{d}(\cdot, \cdot)$ also provides us with an approach to solve the indirect problem. Namely, one can compute the direct distortion $\hat{d}(\cdot, \cdot)$ and compute the RDF for the source Y^N under $\hat{d}(\cdot, \cdot)$.

A. Relevant results

The computation of a direct RDF $R_U(D)$ of a source U over a discrete alphabet \mathcal{U} is performed by minimizing the mutual information over the set of transition probabilities

$$P(\hat{u}|u) \triangleq \mathbb{P}(\hat{U} = \hat{u} | U = u),$$

under the constraint

$$\sum_{u \in \mathcal{U}} \sum_{\hat{u} \in \hat{\mathcal{U}}} Q(u) P(\hat{u}|u) d(u, \hat{u}) \leq D,$$

where $Q(u) \triangleq \mathbb{P}(U = u)$ and $d(\cdot, \cdot)$ is the per-letter distortion measure. This is equivalent to finding a stationary point to the Lagrangian

$$L_0(r, \mathbf{P}) = \sum_{u, \hat{u}} Q(u) P(\hat{u}|u) \left[\log \frac{P(\hat{u}|u)}{\sum_{u, \hat{u}} Q(u) P(\hat{u}|u)} + r(d(u, \hat{u}) - D) \right] \quad (8)$$

over the set of all transition probabilities. By introducing the constraint on the transition probabilities and using the Lagrange dual of (8), Gallager proved in [9] the theorem below.

Theorem II.1. [9, Thm. 9.4.1] *For a given source entropy $H(U)$ and a given distortion measure $d(\cdot, \cdot)$, let*

$$R_0(r, \mathbf{P}) \triangleq \sum_{u, \hat{u}} Q(u) P(\hat{u}|u) \left[\ln \frac{P(\hat{u}|u)}{\sum_u Q(u) P(\hat{u}|u)} + r d(u, \hat{u}) \right],$$

then for any $r > 0$,

$$\min_{\mathbf{P}} R_0(r, \mathbf{P}) = H(U) + \max_{\mathbf{f}} \sum_u Q(u) \ln f_u, \quad (9)$$

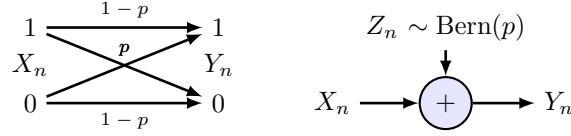


Fig. 2: Equivalent descriptions of the channel $P_{Y|X}$.

where the minimization in the LHS of (9) is over all transition probability functions $\mathbf{P} = \{P(\hat{u}|u), u \in \mathcal{U}, \hat{u} \in \hat{\mathcal{U}}\}$, and the maximization in the RHS of (9) is over all $\mathbf{f} = \{f_u, u \in \mathcal{U}\}$ with non-negative components satisfying the constraints

$$\sum_u f_u e^{-rd(u, \hat{u})} \leq 1, \quad \hat{u} \in \hat{\mathcal{U}}. \quad (10)$$

Necessary and sufficient conditions on \mathbf{f} to achieve the maximum in (9) are the existence of a set of non-negative numbers $\{w(\hat{u}), \hat{u} \in \hat{\mathcal{U}}\}$ satisfying

$$1 = \frac{f_u}{Q(u)} \sum_{\hat{u} \in \hat{\mathcal{U}}} w(\hat{u}) e^{-rd(u, \hat{u})}, \quad (11)$$

and that (10) is satisfied with equality for each \hat{u} with $w(\hat{u}) > 0$.

It follows from (8) that if the conditions for equality in Theorem II.1 hold, we have

$$R_U(D) = \min_{\mathbf{P}} R_0(r, \mathbf{P}) = H(U) + \max_{\mathbf{f}} \sum_u Q(u) \ln f_u.$$

We refer to [10] for a discussion of Theorem II.1 in the context of convex optimization theory as well as a geometric programming representation of this problem.

B. Indirect DRF of a binary i.i.d process

We now specialize our study of the iRDF to the case where X^n is an i.i.d binary process, Y^n is obtained by passing X^n through a memoryless Binary Symmetric Channel (BSC) and for Hamming distortion measure.

More specifically, we focus on the case where $X_i \perp X_j$, $i \neq j$ and

$$Y^n = X^n \oplus Z^n,$$

where X^n and Z^n are two Bernoulli i.i.d process, independent of each other, with $\mathbb{P}(X_i = 1) = \alpha$ and $\mathbb{P}(Z_i = 1) = p$, $\forall i \in \{0 \dots n\}$ respectively. Accordingly, $\mathcal{X} = \mathcal{Y} = \{0, 1\}$ and Y_i is a binary i.i.d process with

$$\beta \triangleq \mathbb{P}(Y_i = 1) = p \star \alpha, \quad \forall i \in \{1 \dots n\}.$$

For the fidelity criterion at the receiver we consider the case $\hat{\mathcal{X}} = \{0, 1\}$ and

$$d(x_i, \hat{x}_i) = x_i \oplus \hat{x}_i, \quad (12)$$

which corresponds to the usual Hamming distance between x^n and \hat{x}^n .

Remark II.2. Given the symmetry in the source X_i and the noisy observations Y_i , we can consider $\alpha, p \leq 1/2$: the remaining cases can be obtained by complementing the observations Y^n and/or the reconstructions \hat{X}^n .

In view of Remark II.2 we will assume $\alpha, p \leq 1/2$ in the remainder of the paper.

III. RESULTS

A. Preliminaries

From the definition of the iRDF we can infer some properties of $R_{X|Y}(D)$ for the model in Fig. 2:

Proposition III.1. The function $R_{X|Y}(D)$ must satisfy the following properties:

- (i) $R_{X|Y}(D) = 0$ for any $D \geq \alpha$.
- (ii) $R_{X|Y}(D)$ is only defined in the interval $D \geq \min\{p, \alpha\}$.
- (iii) $R_{X|Y}(D)$ is non-decreasing in p .

	$y_i = 0$	$y_i = 1$
$\hat{x}_i = 0$	$\frac{\alpha p}{\beta}$	$\frac{\alpha \bar{p}}{\beta}$
$\hat{x}_i = 1$	$\frac{\bar{\alpha} \bar{p}}{\beta}$	$\frac{\bar{\alpha} p}{\beta}$

TABLE I: Possible values of $\hat{d}(y_i, \hat{x}_i)$ in (14)

(iv) $R_{X|Y}(D) \geq R_X(D)$ for any D , where

$$R_X(D) = \begin{cases} h(\alpha) - h(D), & 0 \leq D \leq \alpha, \\ 0, & D > \alpha. \end{cases} \quad (13)$$

is the RDF of X under the Hamming distortion (see e.g. [11]) and corresponds to the case $Y^n = X^n$.

Using the results in Section II, we can equate the indirect RDF $R_{X|Y}(D)$ to the (direct) RDF $R_Y(D)$ by defining the amended distortion measure $\hat{d}(\cdot, \cdot)$ in (6) obtained as

$$\begin{aligned} \hat{d}_n(y^n, \hat{x}^n) &= \sum_{x^n} \sum_{i=1}^n (x_i \oplus \hat{x}_i) \mathbb{P}(X^n = x^n | Y^n = y^n), \\ &= \sum_{i=1}^n \sum_{x_i \in \{0,1\}} (x_i \oplus \hat{x}_i) \mathbb{P}(X_i = x_i | Y_i = y_i) \\ &= \sum_{i=1}^n \mathbb{P}(X_i \neq \hat{x}_i | Y_i = y_i) = \sum_{i=1}^n \hat{d}(y_i, \hat{x}_i). \end{aligned} \quad (14)$$

It follows from (14) that the new distortion measure $\hat{d}(\cdot, \cdot)$ has an intuitive interpretation: if $\hat{x}_i \in \{0, 1\}$ is the estimate of X_i given the symbol $y_i \in \{0, 1\}$, then $\hat{d}(y_i, \hat{x}_i)$ is the probability of making an error in this estimation. Table I lists all the possible values of $\hat{d}(y_i, \hat{x}_i)$.

B. Main Result

The next step is to use Theorem II.1 to derive $R_{X|Y}(D)$.

Theorem III.2. *Let*

$$\begin{aligned} g(r) &\triangleq r(D - p) + \log(1 - e^{-r(u+v)}) \\ &\quad - \bar{\beta} \log(1 - e^{-ru}) - \beta \log(1 - e^{-rv}). \end{aligned} \quad (15)$$

The iRDF $R_{X|Y}(D)$ is given by

$$R_{X|Y}(D) = \begin{cases} h(\beta) - g(r^*) & p \leq D \leq \alpha, \\ 0 & D > \min\{\alpha, p\}, \end{cases}$$

where r^* is the unique solution to

$$\frac{\bar{\beta}u}{e^{r^*u} - 1} + \frac{\beta v}{e^{r^*v} - 1} - \frac{u + v}{e^{r^*(u+v)} - 1} = D - p, \quad (16)$$

with $u \triangleq (\alpha - p)/\beta$ and $v \triangleq (\bar{\alpha} - p)/\bar{\beta}$.

Proof: Only an outline of the proof is provided here: the full proof is provided in App. A. In view of Proposition III.1 it is enough to consider the case $p < D < \alpha \leq 1/2$. Assume that equality holds in (10), then

$$\begin{pmatrix} e^{-r\frac{\alpha p}{\beta}} & e^{-r\frac{\alpha \bar{p}}{\beta}} \\ e^{-r\frac{\bar{\alpha} p}{\beta}} & e^{-r\frac{\bar{\alpha} \bar{p}}{\beta}} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

which implies

$$\begin{aligned} f_0 &= \frac{1 - e^{-ru}}{e^{-r\frac{p\alpha}{\beta}} (1 - e^{-r(u+v)})}, \\ f_1 &= \frac{1 - e^{-rv}}{e^{-r\frac{p\bar{\alpha}}{\beta}} (1 - e^{-r(u+v)})}, \end{aligned}$$

where $u \triangleq (\alpha - p)/\beta$ and $v \triangleq (\bar{\alpha} - p)/\bar{\beta}$. Note that both u and v are positive in the domain of interest. We next write

$$\begin{aligned} R_{X|Y}(D) &\geq h(\beta) + \bar{\beta} \log(1 - e^{-ru}) + \beta \log(1 - e^{-rv}) \\ &\quad - \log(1 - e^{-r(u+v)}) - r(D - p) \\ &= h(\beta) - g(r). \end{aligned} \quad (17)$$

In order to maximize the RHS of (17), we take the derivative of $g(r)$ which gives

$$g'(r) = -(D - p) + \frac{u + v}{e^{r^*(u+v)} - 1} - \frac{\bar{\beta}u}{e^{r^*u} - 1} - \frac{\beta v}{e^{r^*v} - 1}. \quad (18)$$

It can be shown that $\lim_{r \rightarrow \infty} g'(r) = p - D < 0$, $\lim_{r \rightarrow 0^+} g'(r) = 1/2 - D > 0$ and that $g'(r)$ is non-decreasing for $r > 0$. All this implies that the maximum of $g(r)$ is obtained at a single point r^* in the domain $r > 0$ which corresponds to $g'(r^*) = 0$. We conclude that this r^* maximizes the RHS of (17).

It is shown in Appendix A that for $p < \alpha \leq 1/2$ and $r = r^*$, there exist positive w_0 and w_1 that satisfy (11). This implies that substituting r^* in (17) leads to equality, i.e., the iRDF is given by the RHS of (17). ■

In the special case where $\alpha = 1/2$ and $p < \alpha$, we have that $\beta = 1/2$ and (16) reduces to

$$\frac{(\bar{p} - p)}{e^{r(\bar{p} - p)} - 1} - \frac{2(\bar{p} - p)}{e^{2r(\bar{p} - p)} - 1} = D - p, \quad (19)$$

which leads to

$$r^* = \frac{\log\left(\frac{\bar{p} - D}{D - p}\right)}{\bar{p} - p}.$$

Substituting r^* in (15) results in $g(r^*) = h(\Delta)$, where

$$\Delta \triangleq \Delta(D, p) \triangleq \frac{D - p}{\bar{p} - p}.$$

It follows from Theorem III.2 that

$$R_{X|Y}(D) = \begin{cases} \log(2) - h(\Delta), & p < D < 1/2, \\ 0, & D \geq 1/2, \end{cases} \quad (20)$$

which is equivalent to [2, Exc. 3.8]. Equation (20) has a similar form as the direct RDF (13) of a binary i.i.d symmetric process. It is interesting to compare (20) to (13) and to observe how the properties of $R_{X|Y}$ anticipated in Proposition III.1 are expressed in the special case of (20).

- (i) $D = 1/2$ corresponds to $h(\Delta) = h(1/2) = \log(2)$.
- (ii) The domain of $R_{X|Y}(D)$ is $0 \leq \Delta$ or $p \leq D$.
- (iii) Δ is decreasing in p and therefore $R_{X|Y}(D)$ is increasing in p .
- (iv) (20) reduces to (13) for $p = 0$.

The slope of $R_{X|Y}(D)$ is an important parameter since it determines the maximal return in code-rate reduction for each additional distortion unit the system can tolerate. In the range $p \leq D \leq 1/2$, this slope is given by

$$\frac{1}{\bar{p} - p} \log\left(\frac{\bar{p} - D}{D - p}\right). \quad (21)$$

Note that this slope is more steep than the slope of $R_X(D)$, and goes to infinity as p approaches $1/2$ (see Fig. 3). This fact confirms the intuition that an increment in the bit-rate when describing noisy measurements is less effective in reducing distortion as the intensity of the noise increases.

Another interesting factor is the rate at which $R_{X|Y}(D)$ changes with p for a fixed $p \leq D \leq \alpha \leq 1/2$. This rate represents the amount of excess coding needed as a result of increasing uncertainty on the source in order to keep a fixed distortion.

Due to the similarity between (20) and (13), it may be tempting to guess that $R_{X|Y}(D)$ is given in a similar form to (20) even in the case where $\alpha < 1/2$. While an exact solution of (16) is hard to obtain in general, it is possible to obtain the following bound.

Theorem III.3. For any $p, \alpha \in [0, 1]$ and $D \geq p$,

$$R_{X|Y}(D) \leq h(\beta) - h(\Delta), \quad (22)$$

where $\Delta = (D - p)/(1 - 2p)$.

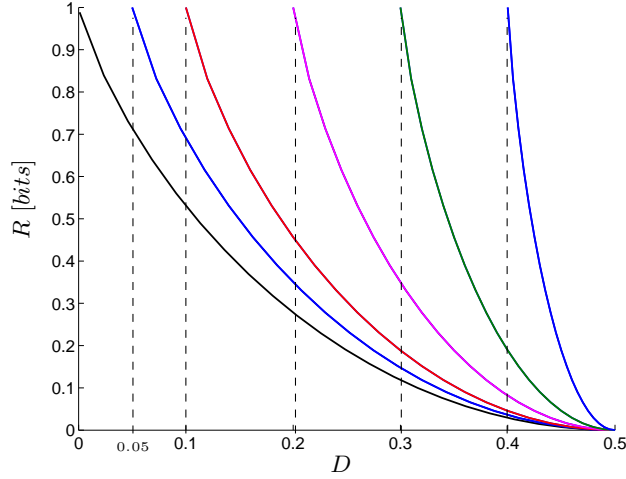


Fig. 3: $R_{X|Y}(D)$ for $\alpha = 1/2$ and various values of $0 \leq p < 1/2$ that correspond to the vertical dashed lines.

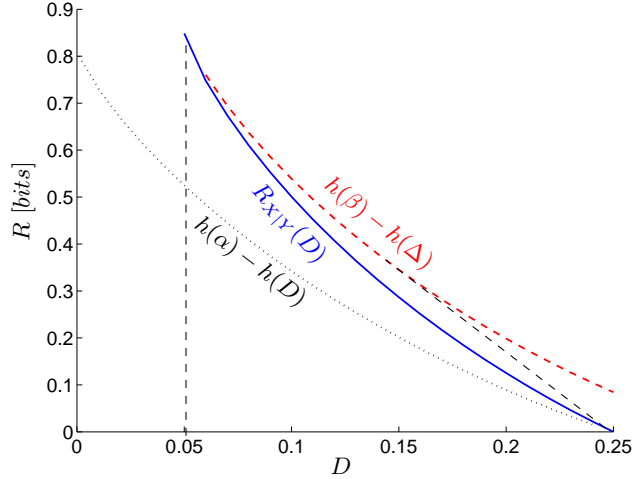


Fig. 4: $R_{X|Y}(D)$, $R_X(D)$ and the upper bound (22) for $\alpha = 1/4$ and $p = 0.05$.

Proof: The proof is provided in App. B. ■

The bound in Theorem III.3 is illustrated in Figure 4. The fact that (22) is not tight in general can be easily seen since $\Delta > \beta$ at $D = \alpha$ for $\alpha \neq 1/2$. In fact, due to the convexity of $R_{X|Y}(D)$, a better bound can be obtained by adding the point $R_{X|Y}(\alpha) = 0$ to the bounding curve and taking the convex closure, as illustrated by the dashed line in Figure 4.

In view of Theorems II.1, III.2 and III.3, the results in this paper can be summarized by the following statement. For $p < D < \alpha$ and any $r > 0$ we have

$$h(\beta) - g(r) \leq R_{X|Y}(D) \leq h(\beta) - h(\Delta), \quad (23)$$

where the LHS holds with equality if and only if r satisfies (16), and the RHS holds with equality if and only if $\alpha = 1/2$.

IV. CONCLUSIONS

This paper studies the indirect rate-distortion problem for a binary i.i.d. source under the Hamming distortion given its noisy observation through a binary symmetric channel. The indirect rate distortion problem is an extension of the rate distortion problem in which the encoder is provided with a noisy observation of the source sequence. We investigate the rate-distortion tradeoff for the simple scenario of a binary source, bit flipping noise and Hamming distortion. Although conceptually simple, this model provides a number of key intuitions on more general models and illustrates important tradeoffs for practical systems. For instance, by deriving the relationship between rate and distortion at each noise level, we make it possible to determine how the sampling error and the communication error probabilities can be balanced in a remote sensor to obtain a desired target end-to-end quality of measurement.

APPENDIX A

In this Appendix we complete the proof of Theorem III.2 by showing the existence of positive w_0 and w_1 that satisfy (11).

From the expression to f_0 and f_1 we obtain:

$$w_0 = \bar{\beta} \frac{e^{-r \frac{\bar{\alpha} p}{\beta}}}{e^{-r \frac{\bar{\alpha} p}{\beta}} - e^{-r \frac{\alpha \bar{p}}{\beta}}} - \beta \frac{e^{-r \frac{\alpha \bar{p}}{\beta}}}{e^{-r \frac{\alpha p}{\beta}} - e^{-r \frac{\alpha \bar{p}}{\beta}}}, \quad (24)$$

$$w_1 = \beta \frac{e^{-r \frac{\alpha p}{\beta}}}{e^{-r \frac{\alpha p}{\beta}} - e^{-r \frac{\alpha \bar{p}}{\beta}}} - \bar{\beta} \frac{e^{-r \frac{\alpha \bar{p}}{\beta}}}{e^{-r \frac{\bar{\alpha} p}{\beta}} - e^{-r \frac{\alpha \bar{p}}{\beta}}}, \quad (25)$$

We need to show that (24) and (25) are positive for any $p < \alpha < 1/2$ and $r = r^*$. The case where $\alpha = 1/2$ were treated above and leads to $w_0 = w_1 = 1/2$. If $p = \alpha$, then it follows from Proposition III.1 that $R_{X|Y}(D)$ is defined only for $D \geq \alpha$ and equals zero. We will therefore assume $p \leq D \leq \alpha < 1/2$. Another way to write (24) and (25) is

$$w_0(r) = \frac{\bar{\beta}}{1 - e^{-ru}} - \frac{\beta e^{-r \frac{\bar{\alpha} \alpha}{\beta \bar{\beta}} (\bar{p} - p)}}{1 - e^{-rv}},$$

$$w_1(r) = \frac{\beta}{1 - e^{-rv}} - \frac{\bar{\beta} e^{-r \frac{\bar{\alpha} \alpha}{\beta \bar{\beta}} (\bar{p} - p)}}{1 - e^{-ru}}.$$

Since $u > 0$, $v > 0$ and $\bar{p} - p > 0$ in the domain of interest, it can be shown that $\lim_{r \rightarrow \infty} w_0(r) = \bar{\beta}$ and that the derivative of $w_0(r)$ is negative for any $r > 0$. This implies that $w_0(r) > 0$ for all values of r in the domain of interest and in particular at $r = r^*$.

For w_1 we can show that $\lim_{r \rightarrow 0^+} w_1(r) = -\infty$, $\lim_{r \rightarrow \infty} w_1(r) = \beta$ and it is monotonically increasing for $r > 0$. By continuity of $w_1(r)$, it follows that there exists $r_0 > 0$ with $w_1(r_0) = 0$ such that $w_1(r) < 0$ whenever $r < r_0$ and $w_1(r) > 0$ whenever $r > r_0$. Since we have seen in the proof of Theorem III.2 that $g'(r)$ has similar behavior with a unique root r^* , we conclude that if $g'(r_0) < 0$, then $r^* > r_0$ and then $w_1(r^*) > 0$. It is therefore enough to show that $g'(r_0) < 0$. Indeed, at $r = r_0$ we have

$$\frac{\beta}{1 - e^{-rv}} = \frac{\bar{\beta} e^{-r \frac{\bar{\alpha} \alpha}{\beta \bar{\beta}} (\bar{p} - p)}}{1 - e^{-ru}}.$$

Substituting that in the expression for $g'(r)$ we obtain

$$b(r) \triangleq g'(r = r_0) = -D + p + \frac{u + v}{e^{r(u+v)} - 1} - \frac{\bar{\beta} u}{e^{ru} - 1} - \frac{\bar{\beta} v e^{-r \left(\frac{\bar{\alpha} \alpha}{\beta \bar{\beta}} (\bar{p} - p) - u \right)}}{e^{ru} - 1}.$$

Define

$$a(r) = -D + p + \frac{u + v}{e^{r(u+v)} - 1} - \frac{\bar{\beta} u}{e^{ru} - 1} - \frac{\bar{\beta} v}{e^{ru} - 1}.$$

Since

$$\frac{\alpha \bar{\alpha}}{\beta \bar{\beta}} (\bar{p} - p) - u > 0,$$

we have that $a(r) > g'(r = r_0)$ for all $r > 0$. In addition, $\lim_{r \rightarrow \infty} a(r) = -D + p < 0$ and

$$a'(r) = (u + v) \left(-\frac{u + v}{(e^{r(u+v)} - 1)^2} + \frac{u \bar{\beta}}{(e^{ru} - 1)^2} \right),$$

which is positive for all $r > 0$. We conclude that $b(r) < a(r) < 0$ for all $r > 0$. This proves the claim.

APPENDIX B

Proof of Th. III.3

It is enough to assume that $p \leq \alpha \leq 1/2$. For $\alpha = 1/2$ we have

$$g(r) = \log \left(1 - e^{-r(\bar{p} - p)} \right) - \log \left(1 - e^{-2r(\bar{p} - p)} \right). \quad (26)$$

We will show that for all $r > 0$, the difference between $g(r)$ that corresponds to any $p \leq D \leq 1/2$ and the one that corresponds to $\alpha = 1/2$ is always positive. This difference can be written as

$$\begin{aligned} \delta(r) \triangleq & \bar{\beta} \log \left(\frac{1 - e^{-r(u+v)}}{(1 - e^{-ru})(1 + e^{-r(\bar{p}-p)})} \right) \\ & + \beta \log \left(\frac{1 - e^{-r(u+v)}}{(1 - e^{-rv})(1 + e^{-r(\bar{p}-p)})} \right). \end{aligned} \quad (27)$$

The result follows by noting that $\lim_{r \rightarrow \infty} \delta(r) = 0$ and the derivative of $\delta(r)$ is strictly positive for any $r > 0$.

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